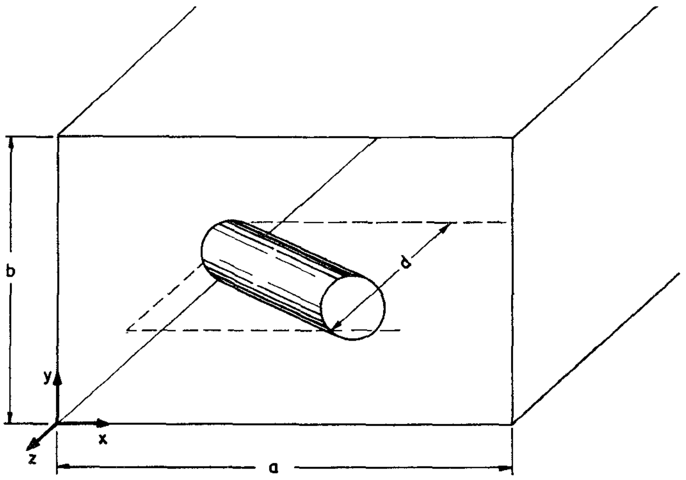

Fig. 1—A dielectric obstacle in a rectangular waveguide. The obstacle is nonsymmetric with respect to a plane perpendicular to the axis z of the waveguide. The cylinder extends a distance d in the z direction.

Fig. 1). In order to obtain bounds on the three distinct elements of $\boldsymbol{B}_\theta$, we evaluate (5) by using three different forms of the trial function, $\boldsymbol{E}_t$. To do this we choose the following three values for e_θ,

$$e_\theta = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\end{pmatrix}. \quad (6)$$

The corresponding values of $e_\theta{}^\dagger \boldsymbol{B}_\theta e_\theta$ are then B_{11}, B_{22}, and $B_{11}+B_{22}+2B_{12}$. The exact solution of a dielectric slab, which fills the region $0 \leq z \leq d$ of the waveguide (the slab completely encloses the obstacle), is introduced as a trial function. The permittivity of the slab is retained as a parameter which can be varied to improve the bounds. The trial function within the region of the dielectric slab is then

$$\boldsymbol{E}_t = \begin{pmatrix} \mathrm{j}C \sin(\pi x/a)\cos Kz \\ \mathrm{j}D \sin(\pi x/a)\sin Kz \end{pmatrix}, \quad (7)$$

where K is the parameter to be varied, C and D are constants and j is a unit vector in the y direction. $e_\theta{}^\dagger \boldsymbol{B}_{\theta t} e_\theta$, and C and D are determined by matching the tangential components $\boldsymbol{E}_t$ in (7) and $\boldsymbol{H}_t$ at $z=d$ to the asymptotic trial expressions ($\boldsymbol{B}_\theta$ replaced by $\boldsymbol{B}_{\theta t}$) of (3), and by specifying the value of θ. It can be shown that for

$$\theta = \pi - kd, \; d(k^2 + W)^{1/2} < \frac{1}{2}\pi$$

we have

$$\beta_\theta \to -\infty$$

and

$$\alpha_\theta > \left[(\pi/2)^2 - k^2d^2 - Wd^2\right]/\rho d^2. \quad (8)$$

The requirement $d(k^2+W)^{1/2} < \frac{1}{2}\pi$ means that the axial extent d of the obstacle must be *less* than $\frac{1}{4}\lambda_g$, where λ_g is the guide wavelength in the dielectric.

Substituting (6), (7) and (8) in (5), we obtain the upper and lower bounds on B_{11}, B_{22}, and $B_{11}+B_{22}+2B_{12}$:

$$-\sec^2(Kd)\left[P^2Q^+ + (2PR + R^2)I_e\right](\alpha_\theta')^{-1}$$
$$\leq Kd \tan(Kd) - KdB_{11}$$
$$+ \sec^2(Kd)(PQ^+ + RI_e) \leq 0, \quad (9a)$$
$$-\csc^2(Kd)\left[P^2Q^- + (2PR + R^2)I_0\right](\alpha_\theta')^{-1}$$
$$\leq -Kd \cot Kd - KdB_{22}$$
$$+ \csc^2(Kd)(PQ^- + RI_0) \leq 0, \quad (9b)$$
$$-\left\{\sec^2(Kd)\left[P^2Q^+ + (2PR + 2R^2)I_e\right]\right.$$
$$+ \csc^2(Kd)\left[P^2Q^- + (2PR + 2R^2)I_0\right]$$
$$\left. + 4(PR + R^2)\sec(Kd)\csc(Kd)I\right\}(\alpha_\theta')^{-1}$$
$$\leq Kd\left[\tan(Kd) - \cot(Kd)\right]$$
$$- Kd(B_{11} + B_{22} + 2B_{12})$$
$$+ \sec^2(Kd)(PQ^+ + RI_e)$$
$$+ \csc^2(Kd)(PQ^- + RI_0)$$
$$+ 2R\sec(Kd)\csc(Kd)I \leq 0, \quad (9c)$$

where

$$P \equiv (kd)^2 - (Kd)^2$$
$$Q^{+,-} \equiv \tfrac{1}{2}\left[1 \pm \sin(2Kd)/(2Kd)\right]$$
$$R \equiv \tfrac{1}{2}Wd^2$$
$$\alpha_\theta' \equiv (\pi/2)^2 - (k^2 + W)d^2$$
$$I_e = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\cos^2(Kz)d\tau$$
$$I_0 = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\sin^2(Kz)d\tau$$
$$I = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\cos(Kz)\sin(Kz)d\tau$$

(b is the narrow dimension of the waveguide). The range of integration in I_0, I_e and I is over the volume of the obstacle.

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Note on Tabulations of Constants for Rigid Hollow Metal Rectangular Waveguide

Precise four decimal place tables of free-space and waveguide wavelength and related ratios for rigid hollow metal rectangular waveguides were computed by the Sperry Microwave Electronics Company and published in this journal in 1956.[1] This set of tables was later extended to cover 28 American waveguide sizes and appeared in a handbook.[2] Booth has published a set of microwave data tables including waveguide wavelength to three decimal places for ten commonly used British rectangular waveguide sizes.[3]

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Unfortunately, two of these tabulations[1,2] use a "low" value for the speed of light: $c=299776$ mks and thus contain errors in their tabulated constants. Booth's tabulations use the presently accepted value of $c=299792.5$ km/sec[4] but cover only ten British waveguide sizes.

The errors entailed by assuming the older "low" value of c can best be explained by examples comparing the free-space and waveguide wavelengths computed using the "low" and "accepted" values of c. Let us first examine the error entailed in the computation of free-space wavelength.

frequency (Gc)	λ (cm) (using $c=299776$ km/sec	λ (cm) (using $c=299792.5$ km/sec
0.275	109.0094	109.0155
1.000	29.9776	29.9793
100.000	0.2998	0 2998

Thus it is seen that for frequencies below 100 Gc/ errors in λ may occur in the third or fourth decimal place tabulated if the "low" value of c is used. In a similar fashion, errors can be observed in waveguide wavelength for a single waveguide size. Let us, for example, examine λ_g for the common 2.000×1.000 inch outside dimension waveguide (IEC R-48, British WG-12, American WR-187 and RG-49/U numbers):

frequency (Gc)	λ_g(cm) (using $c=299776$ km/sec	λ_g(cm) (using $c=299792.5$ km/sec
3.600	17 2416	17.2454
4.800	8.2815	8.2823
6 400	5.3822	5.3825

Errors in λ_g can be observed in the third decimal place tabulated if the "low" value of c is used.

In summary, then, presently available tabulations of rectangular waveguide constants are either slightly restricted in scope or are present numbers that are slightly in error for the most commonly used waveguide sizes for frequencies below 100 Gc.

There are presently available 38 standard rectangular waveguide sizes in approximately two-to-one dimension ratio catalogued according to the International Electrotechnical Commission, American and British systems. To the author's knowledge, no one complete cross referencing of identification systems[5] or complete set of tables

[1] Sperry Microwave Electronics Co., "Tables of constants for rectangular waveguides," IRE Trans. on Microwave Theory and Techniques," vol. MTT-4, 12 page supplement; July, 1956.
[2] "Microwave Engineers' Handbook," Horizon-House-Microwave, Inc., T. S. Saad, Ed., Brookline, Mass.; 1963.
[3] A. E. Booth, "Microwave Data Tables," Iliffe & Sons, London, England; 1959.
[4] A. G. McNish, "The speed of light," IRE Trans. on Instrumentation, vol. I-11, pp. 138–148; December, 1962.
[5] T. N. Anderson, "Waveguide alphabet soup or KXCSLP," *Microwave J.*, vol. 4, pp. 42–43; May, 1961. Cross references American to IEC numbers for 34 American waveguide sizes.)

of constants for all 38 waveguides (using the "accepted value of c) is generally available at the present time.

A complete cross-referenced set of tables has recently been completed at this laboratory using the Frederic (Feranti-Mercury) computer. For each of the 38 waveguide sizes, λ (in cm), λ_g (in cm and inches), $1/\lambda_g$ (in cm^{-1} and $inches^{-1}$), λ_g/λ, and λ/λ_g are tabulated against frequency. Entitled "Intern Rapport E-22, 'Tables of Constants for Thirty-Eight Rigid Hollow Metal Rectangular Waveguides,' 12 November 1963," the complete report is available on request from: *Norwegian Defence Research Establishment, Box 25, Kjeller, Norway.*

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The Use of the Rayleigh-Ritz Method in Nonself-Adjoint Problems

This communication is a comment on the very interesting paper[1] by S. P. Morgan in the May issue of these Transactions. It may be of interest to point out that problems similar to those discussed by Morgan arise in nuclear reactor theory. Here the operators are not, as in footnote[1], complex symmetric integral operators but they are nonself-adjoint, and there is considerable interest in finding their eigenvalues and eigenvectors by Rayleigh-Ritz methods. Morgan is correct, of course, in pointing out that the usual maximum and minimum criteria are lacking in these cases and that there are no bounds or error estimates. However the conclusion that it is impossible to use the methods may be overly pessimistic. The methods have been used[2-5] in reactor theory with considerable success (measured by comparison with exact solutions), and this fact gives hope that they may be useful in laser applications.

The key point in successful use of Rayleigh-Ritz methods is the selection of appropriate "trial" functions. This is an art which is quickly developed by experience and knowledge of the physical process. In the reactor applications one tactic which has been found very effective is to choose trial functions which in a sense "bound" the true eigenfunction. For example, suppose it is known that the true eigenfunction has a peak in the center but it is not known how high the peak is. One would then choose two trial functions, one having a higher and one a lower peak than expected of the eigenfunction. The Rayleigh-Ritz method with the criterion "make stationary" is used to "blend" the two trial functions in the appropriate proportions. The justification for this procedure is basically empirical—it gives good results. However, a theoretical argument has been advanced[6] which tends to make the process somewhat more palatable. The essence of this argument is to view the variational method as a special case of a more general class of approximation methods, the "weighted residual" methods,[7] and then to show that the variational method is, in a certain sense, the best special case within this class.

In the remainder of this communication we outline the variational or Rayleigh-Ritz process in the general nonself-adjoint case and show the connections to the type of operator used in footnote[1].

Let the letters u, v, etc. denote elements of a function space with a complex inner product (u, v), and let L denote a nonself-adjoint linear operator on this space. Then a variational principle for the eigenvalues of L is that the functional

$$F[u, v] = \frac{(u, Lv)}{(u, v)} \tag{1}$$

be stationary with respect to arbitrary independent variations of the argument functions u and v. If $\hat{u}$, $\hat{v}$ denotes the point where F takes on the stationary value λ, then about this point the first variation is

$$\delta F = \frac{1}{(\hat{u}, \hat{v})}\left[(\delta u, [L-\lambda]\hat{v}) + ([L^* - \bar{\lambda}]\hat{u}, \delta v)\right], \tag{2}$$

so that F is stationary if and only if λ, $\hat{v}$ is an eigenpair of L and $\bar{\lambda}$, $\hat{u}$ is an eigenpair of the adjoint operator L^*.

$$L\hat{v} = \lambda\hat{v}; \qquad L^*\hat{u} = \bar{\lambda}\hat{u}. \tag{3}$$

(Here $\bar{\lambda}$ = complex conjugate of λ.)

To apply the Rayleigh-Ritz process we assume approximate solutions in the form

$$\hat{v} \approx \sum_1^n a_j h_j; \qquad \hat{u} \approx \sum_1^n b_i w_i \tag{4}$$

where the w_i, h_j are known functions and the a_j, b_i unknown parameters. Inserting (4) into (1) yields the ratio of bilinear forms

$$\frac{[\bar{b}_1 \cdots \bar{b}_n]\begin{bmatrix} (w_i, Lh_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}{[\bar{b}_1 \cdots \bar{b}_n]\begin{bmatrix} (w_i, h_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}} \tag{5}$$

and requiring this to be stationary yields the matrix eigenvalue problem

$$\begin{bmatrix} (w_i, Lh_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda \begin{bmatrix} (w_i, h_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{6}$$

Now we turn attention to the type of operators considered by Morgan.[1] For these operators the adjoint is just the complex conjugate

$$L^* = \bar{L} \tag{7}$$

(where $\bar{L}$ is the operator such that $\bar{L}u = (\overline{L\bar{u}})$ for all u). For such operators (3) shows that the adjoint eigenfunction is the conjugate of the direct eigenfunction

$$\hat{u} = \bar{\hat{v}}. \tag{8}$$

We may make use of this knowledge to specialize the principle (1) in the following way. For operators satisfying (7) the stationary points of $F[u, v]$ have the "natural" property that they are conjugate pairs of functions. Therefore, if the class of admissible pairs (u, v) be restricted to only conjugate pairs, then the functional over the restricted domain becomes

$$F[\bar{v}, v] = \frac{(\bar{v}, Lv)}{(\bar{v}, v)} \tag{9}$$

and has the same stationary points as the unrestricted functional. The functional (9) is recognized as $R[\varphi]$ [(9) in Morgan[1]] by identifying (u, v) with the integral

$$\int_a^b \bar{u}(x)v(x)dx$$

and Lh with the operation

$$\int_a^b k(x, y)h(y)dy.$$

The Ritz process applied to (9) yields

$$\begin{bmatrix} (\bar{h}_i, Lh_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda \begin{bmatrix} (\bar{h}_i, h_j) \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{10}$$

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[1] S. P. Morgan, "On the integral equations of laser theory," IEEE Trans. on Microwave Theory and Techniques, vol. MTT-11, pp. 191–193; May, 1963.

[2] S. Kaplan, "Some new methods of flux synthesis," *Nucl. Sci. Engrg.*, vol. 13, pp. 22–31; May, 1962.

[3] S. Kaplan, O. J. Marlowe and J. A. Bewick, "Application of synthesis techniques to problems involving time dependence," *Nucl. Sci. Engrg.*, vol. 18, p. 2; February, 1964.

[4] D. S. Selengut, "Variational analysis of multidimensional systems," *Nucl. Phys. Quart. Rept.*, vol. HW-59126, pp. 89–124; January, 1959.

[5] G. P. Calame and F. D. Federighi, "A variational procedure for determining spatially dependent thermal spectra," *Nucl. Sci. Engrg.*, vol. 10, p. 190; 1961.

[6] S. Kaplan, "On the Best Method for Choosing the Weighting Functions in the Method of Weighted Residuals," *Trans. American Nucl. Soc. Mtg.*, Salt Lake City, Utah, vol. 6; June, 1963.

[7] S. H. Crandall, "Engineering Analysis," McGraw-Hill Book Co., Inc., New York, N. Y.; 1956.

Author's Reply

I wish to thank Dr. Kaplan for calling my attention to the use of variational principles for approximating the eigenvalues of nonself-adjoint operators in nuclear reactor theory. In the problems to which he refers, the functions and operators are all real, and it is possible that variational calculations may be more easily justified for real nonself-adjoint operators than for complex ones.

For complex symmetric operators,[1] it is definitely *not* true that the Rayleigh-Ritz procedure leads to the best approximation obtainable with a given set of trial functions, unless the space spanned by the trial functions happens to include the (unknown) exact eigenfunction. Specifically, it does not minimize the distance between the exact and approximate eigenvalues, and it does not minimize the distance between the exact and approximate eigenfunctions, in terms of a quadratic metric.

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